# Ambiguous Language and Common Priors*

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**Abstract**

Standard economic models cannot capture the fact that information is often *ambiguous*, and is interpreted in multiple ways. Using a framework that distinguishes between the language in which statements are made and the interpretation of statements, we demonstrate that, unlike in the case where there is no ambiguity, players may come to have different beliefs starting from a common prior, even if they have received exactly the same information, unless the information is common knowledge.

## 1 Introduction

Natural language is often ambiguous; the same message can be interpreted in different ways by different people.[1] Ambiguous language can lead to misunderstandings, and strategic actors may try to exploit ambiguity to their advantage, for example, when writing contracts (Scott and Triantis, 2006) or communicating policy intentions (Blinder et al., 2008).

Such ambiguity is hard to model using standard models, which do not separate meaning from message. We therefore develop a framework that distinguishes between the language that

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[1] We thus use the term ambiguity in a different sense than the decision-theory literature, where ambiguous events are events that the decision-maker cannot assign a precise probability to.

players use and the interpretation of the terms. The language that players use is common, but players may interpret terms differently. For example, an announcement by the Central Bank that it will "take adequate measures" to support banks in distress can have a different meaning to different parties. More formally, the clause may be true in one set of states of the world for one party, but in an altogether different set of states according to another.

We demonstrate that in the presence of ambiguity, players can have the same beliefs only under very restrictive assumptions. Specifically, we show that even if players have a common prior, receive the same signals, and believe that all players receive the same signals, then their posteriors can differ, unless the content of the common signals is common knowledge. This is true even in the absence of ambiguity: as we show, a seemingly weak condition that ensures that players have identical beliefs when there is no ambiguity is in fact equivalent to the strong condition that the information is commonly known. This suggests that the assumption that signals are unambiguous, that is, admit only a single interpretation, is far from innocuous. One implication of our results is that one of the standard justifications for the common-prior assumption, due to Harsanyi (1968) and Aumann (1987), is not valid. Given the centrality of the common-prior assumption in economics, this reinforces the importance of developing alternative models that do not rely on the common-prior assumption.

A number of authors have considered the role of ambiguity in explaining economic phenomena. Harris and Raviv (1993), for instance, show that speculative trade is possible when traders have a common prior and observe public signals if they use different statistical models to update their beliefs, which they interpret as traders interpreting signals differently. Bernheim and Whinston (1998) study how players can benefit from leaving some obligations ambiguous in contracts. Blume and Board (2014) demonstrate that the strategic use of ambiguous messages can mitigate conflict and may thus be welfare enhancing. Neither of these papers models ambiguity syntactically, as we do, which makes it hard to apply the models to related but different situations. Grant et al. (2009) model contracting in the face of ambiguity. They model ambiguity by assuming that unambiguous terms do not fully specify the state of the world. Board and Chung (2012) use a simplified model of the one presented here to study ambiguous contracts.

Rather than focusing on specific applications, we characterize the implications of ambiguity generally, and illustrate how ambiguity can give more insight into differences in beliefs. Our results imply that the common-prior assumption is hard to justify even if players have the same information and start out with identical priors. They are thus related to, but distinct from, those of Brandenburger et al. (1992), who showed that there is an equivalence between models with heterogeneous beliefs and models in which players make information processing errors. Our results imply that one does not even need to assume information processing errors to obtain heterogeneous beliefs. They can thus be viewed as formalizing a (qualitative) argument by Morris (1995, p. 234), who argues that it can be the case that players have the same information but different beliefs even if players are rational and follow the rules of Bayesian updating.

Our paper appears to be first to focus on the crucial role of language when it comes to ambiguity, and to drop the assumption that ambiguity is commonly understood. In doing so, our paper is one of the very few papers that provides a formal argument that the common-prior assumption can be justified only under very restrictive assumptions.[2]

# 2 Example

We start with an example. There are two players, labeled 1 and 2. Players' information is given by the color of a (particular) car that they observe. As in the standard framework, a state (of the world) $\omega$ describes all relevant (non-epistemic and epistemic) aspects of the situation. For simplicity, suppose the car can either be burgundy, rosewood, or scarlet (all different shades of red). Each color corresponds to a specific wavelength (approximately 403 nanometers, 397 nanometers, and 407 nanometers for the three shades, respectively). While the formula "the wavelength corresponding to the color of the car is 403 nanometers" has an unambiguous interpretation, a statement such as "the color of the car is burgundy" can be interpreted in multiple ways: player 1, for example, may draw the line between burgundy and rosewood at 399 nanometers, while player 2 may do so at 401 nanometers. Likewise, player 1 may draw the line between burgundy and scarlet at 404 nanometers, while player 2 does so at 406 nanometers.

A state $\omega$ thus describes the true wavelength (an unambiguous statement) as well as both players' perception of the color (and possibly other aspects). Saying that the interpretation of burgundy is ambiguous means that the set $[[burg]]_1$ of states in which the car is burgundy according to player 1 may differ from the set $[[burg]]_2$ of states in which the car is burgundy according to player 2. For example, if the wavelength is 400 nanometers at a state $\omega$, then player 1 would say that the car is rosewood, while player 2 would say that the car is burgundy (even if they would agree that the wavelength is 400 nanometers if they were to measure it).

For concreteness, assume that there are three states, $\omega_1, \omega_2$, and $\omega_3$. In state $\omega_1$, the wavelength is 400 nanometers, so that player 1 would say that the car is burgundy, while player 2 would say that the car is rosewood. In state $\omega_2$, the wavelength is 403 nanometers, so both players would say that the car is burgundy. In $\omega_3$, the wavelength is 405 nanometers, so that player 1 would say that the color is scarlet, and player 2 would say that the color is burgundy.

[2] See Morris (1995) and references therein for various philosophical objections to the common-prior assumption. Nyarko (2010) and Hellman and Samet (2012) show that the set of states in which the common-prior assumption holds is small in a measure-theoretic and a topological sense, respectively. Rather than showing that the set of states that satisfies the common-prior assumption is small in some sense, we ask whether players' posteriors can remain the same under realistic scenarios.

There is also an unambiguous proposition $q$: each player thinks $q$ is true in $\omega_1$ and $\omega_2$, and false in $\omega_3$. For example, $q$ could be "the wavelength of the car is between 399 and 404 nanometers."

Players' information is given by the color of the car. This means when a player observes the color of the car (viz., burgundy, rosewood, or scarlet), then he thinks possible all states where he sees that the car is burgundy. For example, if player $i$ observes a burgundy car in state $\omega$, then he thinks possible all states in which he has observed a burgundy car (according to $i$). We denote this event by $[[rec_i(burg)]]_i$. So, when player $i$ sees a burgundy car, he updates his beliefs by conditioning on $[[rec_i(burg_i)]]_i$. While the events $[[rec_i(burg)]]_i$ and $[[burg]]_i$ need not coincide (see Section 3.4 for a discussion), we take the two events to be the same here.

Suppose that the true state is $\omega_2$, so both players 1 and 2 observe that the car is burgundy. Then their beliefs about $q$ differ: player 1 assigns probability 1 to $q$, and player 2 assigns probability $\frac{1}{2}$ to $q$. This is true even though the proposition $q$ is not ambiguous, players have a common prior, and receive exactly the same information.

While this example is, of course, very stylized, we believe it is able to model some features of economic situations that cannot be captured by standard economic models. For example, suppose that the formula $burg$ does not refer to the color of a specific car, but instead says that the U.S. economy is burgeoning; the numbers do not refer to wavelengths, but to macroeconomic indicators (such as the number of jobs added in a given month, measured in thousands). An announcement by the central bank that the U.S. economy is burgeoning may lead one player to deduce that at most 403,000 jobs have been added that month, while another player infers that at least 403,000 jobs have been added. This can lead players to have different beliefs about an unambiguous proposition involving future interest rates.[3]

One might think that players can have different beliefs in this example because, even though they receive the same information, they are not certain that that is the case: both players assign probability $\frac{1}{2}$ in $\omega_2$ to the event that the other player has observed a different color than he has. In Section 4, we show that this is not the case: we demonstrate that in the presence of ambiguity, players can have different beliefs about unambiguous propositions, even if they have a common prior, have the same information, and know that they have the same information, unless there is common knowledge of the information that they receive.

[3]It has been argued that such divergence in expectations reduces the effectiveness of monetary policy (Woodford, 2005). In recent years, the Federal Reserve has indeed taken measures to increase transparency not only of their decision-making process, but also of their interpretation of macroeconomic indicators (Bernanke, 2013). As suggested by the example, some ambiguity will most likely remain.

# 3 Framework

To capture ambiguity, we explicitly model players' language, that is, the syntax. The symbols in this language are given meaning in a semantic model. We discuss the syntax in Section 3.1, and the semantics in Section 3.2. We go on to define ambiguity and information structures in Sections 3.3 and 3.4. Much of this material in this section is taken from our companion paper (Halpern and Kets, 2014), where we study the logic of ambiguity. We have simplified some of the definitions here so as to allow us to better focus on the game-theoretic applications.

## 3.1 Syntax

We start by defining the syntax of a formal language that describes all relevant situations. We want a logic where players use a fixed common language, but each player may interpret formulas in the language differently. We also want to allow the players to be able to reason about (probabilistic) beliefs, so as to be able to study the possibility of heterogeneous beliefs.

There is a finite, nonempty set $N = \{1, \dots, n\}$ of players, and a countable, nonempty set $\Phi$ of primitive propositions, such as "the economic outlook is improving" and "growth remains subdued." Let $\mathcal{L}_n^C$ be the set of formulas that can be constructed starting from $\Phi$, and closing off under the following operations:

- conjunction (i.e., if $\varphi$ and $\psi$ are formulas, then so is $\varphi \wedge \psi$ (read "$\varphi$ and $\psi$"));
- negation (i.e., if $\varphi$ is a formula, then so is $\neg\varphi$ (read "not $\varphi$"));
- the modal operator $CB$ (i.e., if $\varphi$ is a formula, then so is $CB\varphi$ (read "$\varphi$ is common belief"));
- the formation of probability formulas, defined below.

Let $\Phi^*$ be the set of formulas that is obtained from $\Phi$ by closing off under negation and conjunction. That is, $\Phi^*$ consists of all propositional formulas that can be formed from the primitive propositions in $\Phi$. Thus, $\Phi^*$ is the subset of formulas in $\mathcal{L}_n^C$ that do not contain probability formulas or belief operators.

We assume that the language is rich enough so that players can talk about signals. That is, there is a subset $\Sigma \subset \Phi^*$ of *signals*, and if $\sigma \in \Sigma$ is a signal, then there is a primitive proposition $rec_i(\sigma) \in \Phi$ for each player $i \in N$. The intended reading of $rec_i(\sigma)$ is that player $i$ has received signal $\sigma$.

Probability formulas describe players' beliefs, and are constructed as follows. If $\varphi_1, \dots, \varphi_k$ are formulas, and $a_1, \dots, a_k, b \in \mathbb{Q}$, then for $i \in N$,

$$a_1 pr_i(\varphi_1) + \ldots + a_k pr_i(\varphi_k) \geq b$$

is a probability formula. Note that we allow for nested probability formulas. The intended reading of $pr_i(\varphi) = x$ is that player $i$ assigns probability $x$ to a formula $\varphi$.

It will be convenient to use some abbreviations. We take $\varphi \vee \psi$ (read "$\varphi$ or $\psi$") to be the abbreviation for $\neg(\neg\varphi \wedge \neg\psi)$, and $\varphi \Rightarrow \psi$ ("$\varphi$ implies $\psi$") to be the abbreviation for $\neg\varphi \vee \psi$. We use the abbreviation $B_i\varphi$ for the formula $pr_i(\varphi) = 1$ that $i$ believes $\varphi$ (with probability 1), and we use the abbreviation $EB\varphi$ for the formula $\wedge_{i\in N} B_i\varphi$ that all players believe $\varphi$, that is, the formula $\varphi$ is *mutual belief*. We write $EB^1\varphi$ for the formula $EB\varphi$, and for $m \geq 2$, take $EB^m\varphi$ to be an abbreviation for $EB(EB^{m-1}\varphi)$; $EB^m\varphi$ says that $\varphi$ is $m^{\text{th}}$*-order mutual belief*.

## 3.2 Semantics

The intended reading of formulas like $\varphi \vee \psi$ and $CB\varphi$ that we gave above is supposed to correspond to intuitions that we have regarding words like "or" and "common belief." These intuitions are captured by providing a semantic model for the formulas in the language, that is, a method for deciding whether a given formula is true or false.

To model that statements can be ambiguous, we want to allow for the possibility that players interpret statements differently. We build on an approach used earlier (Halpern, 2009; Grove and Halpern, 1993), where formulas are interpreted relative to a player. This means that players can disagree on the meaning of a statement.

More specifically, the semantic model we adopt is an epistemic probability structure. An *(epistemic probability) structure* (over a set of primitive propositions $\Phi$) has the form

$$M = (\Omega, \mu, (\Pi_j)_{j\in N}, (\pi_j)_{j\in N}),$$

where $\Omega$ is the state space, assumed to be countable, and $\mu$ is the *common prior* on $\Omega$ (defined on the power set of $\Omega$). For each player $i \in N$, $\Pi_i$ is a partition of $\Omega$, and $\pi_i$ is an *interpretation* that associates with each state a truth assignment to the primitive propositions in $\Phi$. That is, $\pi_i(\omega)(p) \in \{\mathbf{true}, \mathbf{false}\}$ for all $\omega$ and each primitive proposition $p$, where $\pi_i(\omega)(p) = \mathbf{true}$ means that the primitive proposition $p$ is true in state $\omega$ according to player $i$, and $\pi_i(\omega)(p) = \mathbf{false}$ meaning that $p$ is false in state $\omega$ according to player $i$. Intuitively, $\pi_i$ describes player $i$'s interpretation of the primitive propositions.

Standard models use only a single interpretation $\pi$; this is equivalent in our framework to assuming that $\pi_1 = \cdots = \pi_n$. We call a structure where $\pi_1 = \cdots = \pi_n$ a *common-interpretation structure*. Otherwise, we say that it is a structure with *ambiguity*.

The information partitions describe the information that player $i$ has in each state: every cell in $\Pi_i$ is defined by some information that $i$ received, such as signals or observations of the world. Intuitively, player $i$ receives the same information at each state in a cell of $\Pi_i$. We discuss the information partitions in detail in Section 3.4 below.

As is standard, player $i$'s posterior beliefs are derived from his prior $\mu$ and his information partition $\Pi_i$. A player's posterior is derived from the prior by updating: if we write $\Pi_i(\omega)$ for the cell of the partition $\Pi_i$ containing $\omega$, then player $i$'s posterior belief that an event $E \subset \Omega$ is the case is simply the conditional probability $\mu(E \mid \Pi_i(\omega))$ that $E$ holds given his information, that is, given that the state belongs to $\Pi_i(\omega)$. To ensure that posterior beliefs are well-defined, we assume that each partition element has positive (prior) probability, that is, $\mu(\Pi_i(\omega)) > 0$ for each player $i \in N$ and every state $\omega \in \Omega$.

## 3.3 Capturing ambiguity

We use epistemic probability structures to give meaning to formulas. Since primitive propositions are interpreted relative to players, we must allow the interpretation of arbitrary formulas to depend on the player as well. We write $(M, \omega, i) \models \varphi$ to denote that the formula $\varphi$ is true at state $\omega$ according to player $i$ (that is, according to $i$'s interpretation). We define $\models$, as usual, by induction. We start with the primitive propositions $p \in \Phi$. Suppose that $p$ is the statement "the car is burgundy." Then $(M, \omega, i) \models p$ is true precisely when player $i$ would say that the car is burgundy if he knew the state $\omega$ of the world. That is, player $i$ might not be able to see the car, and thus is uncertain as to whether or not it is burgundy. But if player $i$ saw the car, he would call it burgundy (even if another agent would call it scarlet).

Formally, if $p$ is a primitive proposition,

$$(M, \omega, i) \models p \text{ iff } \pi_i(\omega)(p) = \mathbf{true}.$$

This just says that player $i$ interprets a primitive proposition $p$ according to his interpretation function $\pi_i$. Denote by $[[p]]_i$ the set of states where $i$ assigns the value $\mathbf{true}$ to $p$.

For negation and conjunction, as is standard,

$$(M, \omega, i) \models \neg\varphi \text{ iff } (M, \omega, i) \not\models \varphi,$$
$$(M, \omega, i) \models \varphi \wedge \psi \text{ iff } (M, \omega, i) \models \varphi \text{ and } (M, \omega, i) \models \psi.$$

This immediately fixes the interpretation of disjunction, given that $\varphi \vee \psi$ is just $\neg(\neg\varphi \wedge \neg\psi)$:

$$(M, \omega, i) \models \varphi \vee \psi \text{ iff } (M, \omega, i) \models \varphi \text{ or } (M, \omega, i) \models \psi.$$

We can extend the interpretation function $\pi_i(\cdot)$ to $\Phi^*$ in a straightforward way, and write $[[\varphi]]_i$ for the set of the states of the world where the formula $\varphi \in \Phi^*$ is true according to player $i$.

A critical question is how to interpret probability formulas such as $pr_j(p) \geq b$. We interpret $(M, \omega, i) \models \varphi$ as saying "if $i$ had all the relevant information about the state of the world, player $i$ would say that $p$ is true." To interpret whether $pr_j(p) \geq p$ is true, $i$ would need to know $j$'s interpretation $p$. But this interpretation is provided by the state. Thus, player $i$ uses player $j$'s interpretation in determining whether $pr_j(p) \geq b$ is true. That is,

$$(M, \omega, i) \models a_1 pr_j(\varphi_1) + \ldots + a_k pr_j(\varphi_k) \geq b \text{ iff } a_1\mu_j([[\varphi_1]]_j \mid \Pi_j(\omega)) + \ldots + a_k\mu_j([[\varphi_k]]_j \mid \Pi_j(\omega)) \geq b,$$

where $[[\varphi]]_j$ is the set of states $\omega$ of the world such that $(M, \omega, j) \models \varphi$. Hence, according to player $i$, player $j$ assigns $\varphi$ probability at least $b$ if and only if the set of worlds where $\varphi$ holds according to $j$ has probability at least $b$ according to $j$. Thus, player $i$ "understands" $j$'s probability space, in the sense that $i$ uses $j$'s partition element $\Pi_j(\omega)$ and $j$'s probability measure $\mu_j$ in assessing the probability that $j$ assigns to each event.

Given our interpretation of probability formulas, the interpretation of $B_j\varphi$ and $EB^k\varphi$ follows immediately:

$$(M, \omega, i) \models B_j\varphi \text{ iff } \mu_j([[\varphi]]_j \mid \Pi_j(\omega)) = 1,$$

and

$$(M, \omega, i) \models EB\varphi \text{ iff } \mu_j([[\varphi]]_j \mid \Pi_j(\omega)) = 1 \text{ for all } j \in N.$$

It is important to note that $(M, \omega, i) \models \varphi$ does not imply $(M, \omega, i) \models B_i\varphi$: while $(M, \omega, i) \models \varphi$ means "$\varphi$ is true at $\omega$ according to $i$'s interpretation," this does not mean that $i$ believes $\varphi$ at state $\omega$. The reason is that $i$ can be uncertain as to which state is the actual state. For $i$ to believe $\varphi$ at $\omega$, $\varphi$ would have to be true (according to $i$'s interpretation) at all states to which $i$ assigns positive probability. Finally, we define

$$(M, \omega, i) \models CB\varphi \text{ iff } (M, \omega, i) \models EB^k\varphi \text{ for } k = 1, 2, \ldots.$$

If all players interpret a formula $\psi$ in the same way in a given structure $M$ (i.e., for any $i, j \in N$, $(M, \omega, i) \models \psi$ if and only if $(M, \omega, j) \models \psi$), we sometimes write $(M, \omega) \models \psi$ for $(M, \omega, \ell) \models \psi$ (where, of course, player $\ell$ can be chosen arbitrarily). If $\varphi$ is a probability formula or a formula of the form $CB\varphi'$, then it is easy to see that all players interpret $\varphi$ the same way: $(M, \omega, i) \models \varphi$ if and only if $(M, \omega, j) \models \varphi$.

## 3.4 Information structure

Information partitions are generated by signals, which are truthful but may be ambiguous. That is, players receive information, or signals, about the true state of the world, in the form of strings (formulas). Each player understands what signals she and other players receive in different states of the world, but players may interpret signals differently.

Formally, in each state of the world $\omega$, each player $i$ receives a signal $\sigma_{i,\omega} \in \Sigma$ that determines the states of the world he thinks possible. That is, for each player $i \in N$ and every state $\omega \in \Omega$, we have $\Pi_i(\omega) = [[rec_i(\sigma_{i,\omega})]]_i$, where we recall that the intended reading of $rec_i(\sigma_{i,\omega}) \in \Phi$ is that "$i$ received signal $\sigma_{i,\omega}$." We take a broad view of signals here. Getting a message saying $\sigma$ can count as receiving $\sigma$; so can making an observation of $\sigma$ (such as observing that the car is burgundy). For each $i \in N$, the collection $\{[[rec_i(\sigma_{i,\omega})]]_i : \omega \in \Omega\}$ is a partition of $\Omega$, and we assume that for every state $\omega \in \Omega$, $\omega \in [[rec_i(\sigma_{i,\omega})]]_i$.[4] Different players may receive different signals in state $\omega$; moreover, the formula $rec_i(\sigma_{i,\omega})$ may be interpreted differently by each player. We assume that player $j$ understands that player $i$ may be using a different interpretation than he does, so that $j$ correctly infers that the set of states that $i$ thinks are possible in $\omega$ is $\Pi_i(\omega) = [[rec_i(\sigma_{i,\omega})]]_i$.[5] Note that in a state where $rec_i(\sigma_{i,\omega})$ is true according to player $i$, $B_i(\sigma_{i,\omega})$ is also true. However, as we observed in the burgundy car example in Section 2, while the interpretation of $B_i(\sigma_{i,\omega})$ is player-independent, like that of all formulas of the form $B_i\psi$, the interpretation of $rec_i(\sigma_{i,\omega})$ may depend on the player.

So, our framework presents a minimal departure from the standard model: while players can interpret formulas differently, they understand that there is ambiguity, and they even understand each other's interpretation. In the next section, we show that even under such a minimal deviation, the standard justification for the common-prior assumption is problematic.

## 4 Understanding differences in beliefs

Harsanyi (1968, p. 497) writes that "discrepancies among the (prior) probability distributions will themselves often admit of explanation in terms of differences in the information available to different players." Similarly, Aumann (1987) has argued that "people with different information may legitimately entertain different probabilities, but there is no rational basis for people who have always been fed precisely the same information to do so." Here we show that this is no longer true when information is ambiguous, even if players have a common prior and fully understand the ambiguity that they face, unless certain strong assumptions on players' beliefs about the information that others receive are satisfied.

To formalize the argument of Harsanyi and Aumann, we focus on the case where players receive the same signal in a given state. Formally, we say that a propositional formula $\sigma_\omega \in \Sigma$ is a *common signal* at $\omega$ if $\sigma_{i,\omega} = \sigma_\omega$ for all $i \in N$. As noted earlier, players may interpret the event that a given player receives a signal differently, that is, we may have $[[rec_1(\sigma_\omega)]]_i \neq$

[4] It is not hard to see that, in fact, the partition $\Pi_i$ consists of all sets of the form $[[rec_i(\sigma)]]_i$ for $\sigma \in \Sigma$.

[5] We allow disagreement on whether a player has received a particular signal (i.e., we may have $[[rec_i(\sigma)]]_i \neq [[rec_i(\sigma)]]_j$ for some signal $\sigma$) even if players agree on the interpretation of the signal (i.e., $[[\sigma]]_i = [[\sigma]]_j$). However, our results do not rely on this distinction. What we need for our results is that players can draw different conclusions from receiving a signal. Whether that stems from differences in interpreting the signal (as in Section 2) or differences in interpreting what it means to receive a signal is immaterial. Distinguishing between $\sigma$ and $rec_i(\sigma)$ is important for proper conditioning, however (see, e.g., Grünwald and Halpern (2003)).

$[[rec_1(\sigma_\omega)]]_j$. In addition, it may be that some player thinks possible states where the other players have received a signal other than $\sigma_\omega$, so that he does not know that the signal is in fact common to all players. Similarly, a player may think possible states where another player thinks possible states where other players have received a signal other than $\sigma_\omega$, and so on.

One natural formalization of the condition that players are always "fed the same information" is that players receive a common signal. The example in Section 2 demonstrates that if there is ambiguity and players are merely fed the same information, then posteriors need not coincide; the next example demonstrates that this is true even if players have the same interpretation.

**Example 4.1.** There are two players, 1 and 2, and two states, labeled $\omega$ and $\omega'$. The common prior $\mu$ gives each state equal probability, and players have the same interpretation. The propositional formula $\sigma_\omega \in \Sigma$ is a common signal in $\omega$; and players' information is given by $\Pi_1(\omega) = [[rec_1(\sigma_\omega)]]_1 = \{\omega\}$ and $\Pi_2(\omega) = [[rec_2(\sigma_\omega)]]_2 = \{\omega, \omega'\}$. In state $\omega$, both players receive signal $\sigma_\omega$, so that $(M, \omega) \models rec_1(\sigma_\omega) \wedge rec_2(\sigma_\omega)$. However, players 1 and 2 assign different probabilities to the event $E = \{\omega\}$. $\triangleleft$

So, assuming that players receive the same information is not sufficient for posteriors to coincide. In Example 4.1, players receive a common signal in state $\omega$, but player 2 thinks it is possible that player 1 has received a different signal. A natural strengthening of the condition that players are fed the same information is that players receive a common signal *and* believe that all players have received a common signal. In that case, we say that the signal is shared. Formally, a common signal $\sigma_\omega$ at state $\omega$ is a *shared signal* at $\omega$ if

$$(M, \omega) \models EB\big(\wedge_{i \in N} rec_i(\sigma_\omega)\big).$$

The next result (whose proof, like all others, we defer to the appendix) shows that when there is no ambiguity, it suffices for players to receive a shared signal for

**Proposition 4.2.** Suppose $M$ is a common-interpretation structure and that $\sigma_\omega \in \Sigma$ is a shared signal at $\omega$. Then players' posteriors are identical at $\omega$: for all $i, j \in N$ and $E \subset \Omega$,

$$\mu_i(E \mid \Pi_i(\omega)) = \mu_j(E \mid \Pi_j(\omega)).$$

So, when there is no ambiguity, receiving shared signals is sufficient for players' posteriors to coincide. However, the next example demonstrates that this is no longer the case if players may interpret formulas differently.

**Example 4.3.** As in the previous example, there are two players, 1 and 2, and two states, labeled $\omega$ and $\omega'$. The common prior gives each state equal probability. Each player believes that the other player receives the signal $\sigma$ if and only if she herself does:

$$[[rec_1(\sigma)]]_1 = [[rec_2(\sigma)]]_1 = \{\omega\};$$
$$[[rec_1(\sigma)]]_2 = [[rec_2(\sigma)]]_2 = \{\omega, \omega'\}.$$

Let $\Pi_i(\omega) = [[rec_i(\sigma)]]_i$, for $i = 1, 2$. Note that this is not a common-interpretation structure. Nevertheless, $\sigma$ is a shared signal at $\omega$, that is,

$$(M, \omega) \vDash EB(rec_1(\sigma) \wedge rec_2(\sigma)).$$

But the posteriors differ: player 1 assigns probability 1 to $\omega$, and player 2 assigns probability $\frac{1}{2}$ to $\omega$. ◁

The problem with Example 4.3 is that, although the signal is shared, the players don't interpret receiving the signal the same way. That is, it is not necessarily the case that player 1 received $\sigma$ from player 1's point of view if and only if player 2 received $\sigma$ from player 2's point of view.

One way of strengthening the condition that all players believe that each player has received the common signal is to require that all players believe that each player has received the common signal, all players believe that all players believe that, and so on, that is, it is common belief that each player has received the common signal. That is, a common signal $\sigma_\omega$ at $\omega$ is a *public signal* at $\omega$ if

$$(M, \omega) \vDash CB\big(\wedge_{i \in N} rec_i(\sigma_\omega)\big).$$

The signal in Example 4.3 is shared, but not public: while player 2 believes at $\omega$ that both have received the common signal $\sigma_\omega$, she does not believe that player 1 believes that: she assigns probability $\frac{1}{2}$ to the state $\omega'$, in which player 1 believes that neither player has received $\sigma_\omega$.

The next result shows that players' posteriors coincide if and only if they receive a public signal.

**Proposition 4.4.** If $\sigma$ is a public signal at $\omega$, then players' posteriors are identical at $\omega$: for all $i, j \in N$ and $E \subset \Omega$,

$$\mu_i(E \mid \Pi_i(\omega)) = \mu_j(E \mid \Pi_j(\omega)).$$

Conversely, if players receive a common signal $\sigma$ at $\omega$, and players' posteriors are identical at $\omega$, then $\sigma$ is a public signal at $\omega$.

So, unless signals are public, players can have different beliefs about unambiguous statements, even if everybody receives the same signal and knows that everybody has received the same signal. The assumption that players receive a public signal is, of course, very strong: players receive a common signal, and it is commonly believed that they receive that signal.

While the conditions for players' posteriors to coincide seems weaker when there is no ambiguity, it is in fact equally strong: the next result shows that for common-interpretation structures, a signal is shared if and only if it is public.

**Proposition 4.5.** If $\sigma$ is a common signal at $\omega$, then the following are equivalent:

- $(M, \omega) \models EB(\wedge_j rec_j(\sigma))$; and
- $(M, \omega) \models CB(\wedge_j rec_j(\sigma))$.

Example 4.3 shows that Proposition 4.5 does not hold when there is ambiguity; in that example, the signal $\sigma$ is shared, but it is not public.

To summarize, in an environment where players have a common prior and receive information in the form of common signals, the conditions for players to have identical beliefs are very strong: posterior beliefs coincide if and only if signals are public.

# 5 Concluding remarks

While it has been widely recognized that language or, more generally, signals can be ambiguous, existing game-theoretic models cannot capture this in a natural way. In the standard model, information structures can be extremely general, making it possible to model almost any situation. This generality is also a weakness, however: it is not a priori clear what restrictions to impose on beliefs to capture a phenomenon such as ambiguity in meaning. We circumvent this problem by modeling ambiguity directly, using a formal logic. This automatically delivers the restrictions on beliefs that we need when modeling settings with ambiguous signals. In fact, as we show in the appendix, allowing ambiguity is, in a precise sense, less permissive than allowing heterogeneous priors. Since not every heterogeneous prior can be explained by ambiguity, ambiguity meets the criterion of Morris (1995), who argued that "[n]ot *any* heterogeneous prior beliefs should be acceptable as explanations. We should resort to unmodelled heterogeneities in prior beliefs only when we can imagine an origin for the differences in beliefs." Ambiguity can be viewed as providing an origin for differences of beliefs, and also provides a limitation on the types of beliefs allowed.

Ambiguity may also suggest plausible ways of weakening the common prior assumption. Consider again the example in Section 2. Rather than each player receiving a signal about the state of the economy, suppose that there is a public announcement by the Central Bank. Suppose for simplicity that (it is common knowledge that) the announcement will say either "the economy is burgeoning" or "the economy is not burgeoning." To capture this, we need to expand the state space. Suppose that both players believe that the announcement is truthful. Thus, player 1 believes that in state $\omega_1$, the announcement will be "the economy is burgeoning" while player 2 believes it will be "the economy is not burgeoning"; there is similar disagreement about what the announcement will be in state $\omega_3$. To model this, we need to replace the states $\omega_1$ and $\omega_3$ by $\omega_1^+$, $\omega_1^-$, $\omega_3^+$, $\omega_3^-$, where, for $k \in \{1, 3\}$, the announcement is "the economy is burgeoning" in $\omega_k^+$ and "the economy is not burgeoning" in $\omega_k^-$. Since the partition is determined by the observation, which in this case is public, both players' partitions consist of $\{\omega_1^+, \omega_2, \omega_3^+\}$ and $\{\omega_1^-, \omega_3^-\}$. However, since players believe that the announcement is truthful, player 1 (who thinks that the economy is burgeoning in $\omega_1$, but not in $\omega_3$), assigns prior probability 0 to $\omega_1^-$, while he assigns positive prior probability to $\omega_3^-$. For player 3, who thinks that the economy is burgeoning in $\omega_3$ but not in $\omega_1$, it is the reverse. So, after hearing that the economy is burgeoning, player 1 puts probability $\frac{1}{2}$ on each of $\omega_1^+$ and $\omega_2$, while player 2 puts probability $\frac{1}{2}$ on each of $\omega_2$ and $\omega_3^+$. So, even though players have a common prior before the announcement, they do not have a common prior on the expanded state space, and players can agree to disagree. It is easy to show that speculative trade is possible in this environment, even though all announcements are public, and there is no private information; see Harris and Raviv (1993) for a model along these lines that can explain central features of trading patterns.

In formulating our logic, we have stayed as close as possible to the standard framework, by assuming that players fully understand the ambiguity that they face: they understand that others may interpret a statement differently than they do, even if they are uncertain as to the precise interpretation of the other players. This minimal departure from the standard framework already allows for new insights, such as the finding that the common-prior assumption can be expected to hold only under fairly restrictive circumstances. An interesting question for future research is how game-theoretic predictions change when we vary the level of sophistication of players, from the fully sophisticated ones studied here, to players who understand that there is ambiguity, but do not understand that others understand that, and so on, to the fully unsophisticated, who are unaware of any ambiguity.

# A Proofs

## A.1 Proof of Proposition 4.2

Suppose that $(M, \omega) \models EB(\wedge_{i,j \in N} rec_i(\sigma_\omega))$, and that $\Pi_i(\omega) = [[rec_i(\sigma_\omega)]]_i$ for all $i \in N$. We first show that $\mu(\Pi_i(\omega)) = \mu(\Pi_i(\omega) \cap \Pi_j(\omega))$ for all players $i, j \in N$. Let $i \in N$. Then, by assumption,

$$\mu(\{\omega' : (M, \omega') \models \wedge_j rec_j(\sigma_\omega)\} \mid \Pi_i(\omega)) = 1,$$

and it follows that

$$\begin{aligned} \mu(\{\omega' : (M, \omega') \models rec_j(\sigma_\omega)\} \mid \Pi_i(\omega)) &= \mu(\Pi_j(\omega) \mid \Pi_i(\omega)) \\ &= 1. \end{aligned}$$

By the definition of conditional probability, we thus have $\mu(\Pi_i(\omega)) = \mu(\Pi_i(\omega) \cap \Pi_j(\omega))$.

Since $\mu(\Pi_i(\omega) \cap (\Omega \setminus \Pi_j(\omega))) = 0$, it easily follows that, for all events $E \in \mathcal{F}$, $\mu(E \cap \Pi_i(\omega)) = \mu(E \cap \Pi_i(\omega) \cap \Pi_j(\omega))$. Since we also have $\mu(E \cap \Pi_j(\omega)) = \mu(E \cap \Pi_i(\omega) \cap \Pi_j(\omega))$, it follows that $\mu(E \cap \Pi_i(\omega)) = \mu(E \cap \Pi_j(\omega))$. Moreover, taking $E = \Omega$, we have that $\mu(\Pi_i(\omega)) = \mu(\Pi_j(\omega))$. Thus,

$$\mu(E \mid \Pi_i(\omega)) = \frac{\mu(E \cap \Pi_i(\omega))}{\mu(\Pi_i(\omega))} = \frac{\mu(E \cap \Pi_j(\omega))}{\mu(\Pi_j(\omega))} = \mu(E \mid \Pi_j(\omega)),$$

and the result follows. $\square$

## A.2 Proof of Proposition 4.4

To prove the first claim, we first show that for all $i, j \in N$, we have $\mu(\Pi_j(\omega) \mid \Pi_i(\omega)) = 1$. To see this, note that $(M, \omega) \models CB(\wedge_{\ell \in N} rec_\ell(\sigma))$ implies that $(M, \omega) \models B_i(rec_j(\sigma))$ for all $i, j \in N$. For all $i, j \in N$ and $\omega'$ in the support of $\mu(\cdot \mid \Pi_i(\omega))$, we thus have $(M, \omega') \models rec_j(\sigma)$. In other words, the support of $\mu(\cdot \mid \Pi_i(\omega))$ is contained in $\Pi_j(\omega)$, so that $\mu(\Pi_j(\omega) \mid \Pi_i(\omega)) = 1$. The rest of the proof is now analogous to the proof of Proposition 4.2, and therefore omitted.

We next consider the second claim. As posteriors coincide, we have $\mu(\Pi_j(\omega) \mid \Pi_i(\omega)) = 1$ for all $i, j \in N$, so $\mu(\bigcap_j \Pi_j(\omega) \mid \Pi_i(\omega)) = 1$. Consequently, for all $i \in N$, $\omega' \in \Pi_i(\omega)$, we have that $(M, \omega') \models EB(\wedge_j rec_j(\sigma))$. For $k > 1$, suppose, inductively, that for all $i \in N$, $\omega' \in \Pi_i(\omega)$, we have that $(M, \omega') \models EB^{k-1}(\wedge_j rec_j(\sigma))$. Again, $\mu(\Pi_j(\omega) \mid \Pi_i(\omega)) = 1$ for all $i, j \in N$, it follows that $(M, \omega') \models EB^k(\wedge_j rec_j(\sigma))$ for all $i \in N$ and $\omega' \in \Pi_i(\omega)$. Consequently, $(M, \omega) \models CB(\wedge_j rec_j(\sigma))$. $\square$

## A.3 Proof of Proposition 4.5

Clearly, if $(M, \omega) \models CB(\wedge_j rec_j(\sigma))$, then $(M, \omega) \models EB(\wedge_j rec_j(\sigma))$. So suppose that $(M, \omega) \models EB(\wedge_j rec_j(\sigma)$. Hence, there is some $\omega' \in \Omega$ such that $(M, \omega') \models \wedge_j rec_j(\sigma)$.

Let $i \in N$, and suppose $(M, \omega') \models \wedge_j rec_j(\sigma)$. The first step is to show that $(M, \omega') \models B_i(\wedge_j rec_j(\sigma))$. As $(M, \omega') \models \wedge_j rec_j(\sigma)$, we have that $(M, \omega') \models rec_i(\sigma)$, so $\omega' \in \Pi_i(\omega)$. Since $(M, \omega) \models B_i(\wedge_j rec_j(\sigma))$, we have that

$$\mu(\{\omega'' : (M, \omega'') \models \wedge_j rec_j(\sigma)\} \mid \Pi_i(\omega')) = \mu(\{\omega'' : (M, \omega'') \models \wedge_j rec_j(\sigma)\} \mid \Pi_i(\omega)) = 1.$$

It follows that $(M, \omega') \models B_i(\wedge_j rec_j(\sigma))$. Hence, for all $i \in N$,

$$\{\omega' : (M, \omega') \models \wedge_j rec_j(\sigma)\} \subseteq \{\omega' : (M, \omega') \models B_i(\wedge_j rec_j(\sigma))\},$$

and it follows that

$$(M, \omega) \models EB^2(\wedge_j rec_j(\sigma)).$$

For $k > 0$, suppose that for all $i \in N$ and $\ell \leq k-1$,

$$\big\{\omega' : (M, \omega') \vDash EB^{\ell}\big(\wedge_j rec_j(\sigma)\big)\big\} \subseteq \big\{\omega' : (M, \omega') \vDash B_i\big(EB^{\ell}(\wedge_j rec_j(\sigma))\big)\big\},$$

and it follows that

$$(M, \omega) \vDash EB^{\ell+1}(\wedge_j rec_j(\sigma)).$$

Let $i \in N$ and suppose that $(M, \omega') \vDash EB^k(\wedge_j rec_j(\sigma))$. We want to show that $(M, \omega') \vDash B_i(EB^k(\wedge_j rec_j(\sigma)))$. Since $(M, \omega') \vDash EB^k(\wedge_j rec_j(\sigma))$, we have that $(M, \omega') \vDash B_i(EB^{k-1}(\wedge_j rec_j(\sigma)))$, so that

$$\mu(\{\omega'' : (M, \omega'') \vDash EB^{k-1}(\wedge_j rec_j(\sigma))\} \mid \Pi_i(\omega')) = 1.$$

Hence, by the induction hypothesis,

$$\mu(\{\omega'' : (M, \omega'') \vDash EB^{k}(\wedge_j rec_j(\sigma))\} \mid \Pi_i(\omega')) = 1,$$

and $(M, \omega') \vDash B_i(EB^k(\wedge_j rec_j(\sigma)))$. It follows that for all $i \in N$,

$$\{\omega' : (M, \omega') \vDash EB^k(\wedge_j rec_j(\sigma))\} \subseteq \{\omega' : (M, \omega') \vDash B_i(EB^k((\wedge_j rec_j(\sigma)))\},$$

so that

$$(M, \omega) \vDash EB^{k+1}(\wedge_j rec_j(\sigma)).$$

Consequently, $(M, \omega) \vDash CB(\wedge_j rec_j(\sigma))$, that is, $\sigma$ is a public signal at $\omega$. $\square$

## A.4 Example: Ambiguity is less permissive than heterogeneous priors

We present an example that shows that allowing for ambiguity is strictly less permissive than allowing for heterogeneous beliefs. To prove the result, we need to generalize the class of structures to include structures with heterogeneous priors. Define an (epistemic probability) structure to be a tuple

$$M = (\Omega, (\mu_j)_{j \in N}, (\Pi_j)_{j \in N}, (\pi_j)_{j \in N}),$$

where $\mu_j$ is the prior of player $j$. We can have $\mu_j \neq \mu_i$ for $i \neq j$; if players have a common prior, that is, $\mu_j = \mu_i$ for all players $i$ and $j$, then we are back in the framework of Section 3. We say that a formula $\varphi$ is *valid* in a structure $M$, written $M \vDash \varphi$, if $(M, \omega) \vDash \varphi$ for all states $\omega$ in $M$. Two structures $M$ and $M'$ are *equivalent* if the same formulas are valid both; that is, $M \vDash \phi$ iff $M' \vDash \phi$. (This notion of equivalence is explored in greater depth in Halpern and Kets (2014).)

We now construct a structure $M$ with heterogeneous priors for which there is no equivalent ambiguous structure that satisfies the common-prior assumption. The structure $M$ has three players, one primitive proposition $p$, and two states, $\omega_1$ and $\omega_2$. In $\omega_1$, $p$ is true according to all players; in $\omega_2$, the proposition is false according to all players. Player 1 knows the state: his information partition is $\Pi_1 = \{\{\omega_1\}, \{\omega_2\}\}$. The other players have no information on the state, that is, $\Pi_i = \{\{\omega_1, \omega_2\}\}$ for $i = 2, 3$. Player 2 assigns probability $\frac{2}{3}$ to $\omega_1$, and player 3 assigns probability $\frac{3}{4}$ to $\omega_1$. Hence, $M$ is a common-interpretation structure with heterogeneous priors. We claim that there is no equivalent structure $M'$ that satisfies the common-prior assumption.

To see this, suppose that $M'$ is an equivalent structure that satisfies the common-prior assumption, with a common prior $\nu$ and a state space $\Omega'$. Because $M$ and $M'$ are equivalent, we must have $M' \vDash pr_2(p) = \frac{2}{3}$ and $M' \vDash pr_3(p) = \frac{3}{4}$, and therefore

$$\nu(\{\omega' \in \Omega' : (M', \omega', 2) \vDash p\}) = \tfrac{2}{3}, \tag{A.1}$$
$$\nu(\{\omega' \in \Omega' : (M', \omega', 3) \vDash p\}) = \tfrac{3}{4}. \tag{A.2}$$

Note that $M \vDash B_2(p \Leftrightarrow B_1 p) \wedge B_3(p \Leftrightarrow B_1 p)$. Thus, since $M$ and $M'$ are equivalent, we must have that the same formula is valid in $M'$, i.e., that

$$M' \vDash B_2(p \Leftrightarrow B_1 p) \wedge B_3(p \Leftrightarrow B_1 p). \tag{A.3}$$

But the interpretation of a formula of the form $B_i\psi$ does not depend on the player, so if we define $E = \{\omega' \in \Omega' : (M', \omega', 1) \vDash B_1 p\}$, then (A.1)–(A.3) imply that we must have $\nu(E) = 2/3$ and $\nu(E) = 3/4$, a contradiction.

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