Online Appendix to "The Value of a Coordination Game"

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This appendix contains some results not included in Kets, Kager, and Sandroni (2021), "The Value of a Coordination Game." Unless stated otherwise, all references to sections, results, etcetera, are to Kets et al. (2021).

Appendix I and Appendix II show that the results in Sections 3.3.1–3.3.2 are robust to alternate specifications of the economic problem. Appendix III shows that introspective equilibrium is asymptotically stable except in knife-edge cases.

I Investment

This appendix shows that our results in Section 3.3.1 continue to hold if the investment subsidy is replaced by an investment "bonus" that players receive when they successfully invest. Formally, consider the game form $\boldsymbol{u}^b = (u_{11} + b, u_{12}, u_{21}, u_{22})$, where $b \geq 0$ is the bonus. The following result characterizes the conditions under which miscoordination is more costly than coordination failure in the sense that the value decreases as we move from the regime with coordination failure to the regime with miscoordination (i.e., as ρ falls to $\overline{\rho}$) when the bonus increases.

Theorem I.1 (Investment Bonus). *Fix an introspective type space that satisfies Assumptions 1–5 and is such that there is a positive probability of investment at $\overline{\rho}$ (i.e., $\overline{\tau} < 1$). Suppose there is coordination failure if there is no investment bonus (i.e., $\rho > \overline{\rho}$). Then, the value strictly decreases with the investment bonus b as it induces miscoordination (i.e., the dominance parameter falls to $\overline{\rho}$) if and only if the off-diagonal payoffs are sufficiently small and ρ is not too high. That is, there is a $\rho^c \in (\overline{\rho}, 1)$ such that for all u_{11} and u_{22} with $u_{11} \geq u_{22}$ and for all $\rho > \overline{\rho}$, there exist u_{12}^* and u_{21}^* such that the following holds: For any game form $\boldsymbol{u}^b = (u_{11} + b, u_{12}, u_{21}, u_{22})$ with dominance parameter ρ at $b = 0$, as b increases, the value falls below u_{22} at $\overline{\rho}$ if and only if $\rho < \rho^c$, $u_{12} < u_{12}^*$, and $u_{21} < u_{21}^*$.*

Proof. Let $(\overline{p_{11}}, \overline{p_{12}}, \overline{p_{21}}, \overline{p_{22}})$ be the probability distribution over action profiles in introspective equilibrium for the given introspective type space when the dominance parameter is $\overline{\rho}$. Recall that this distribution depends only on the type space. Define the dominance parameter ρ^c by

$$\rho^c := \overline{\rho}\,\frac{\overline{p_{11}} + \overline{p_{21}}}{\overline{p_{11}}}.$$

Clearly, since $\overline{p_{11}}$ and $\overline{p_{21}}$ are strictly positive, we have $\rho^c > \overline{\rho}$. Moreover, it follows from Eq. (8) in the proof of Proposition 3.1 that $\rho^c < 1$. Let $\boldsymbol{u}^b = (u_{11} + b, u_{12}, u_{21}, u_{22})$, where $b \geq 0$. Suppose that in the absence of a bonus there is coordination failure in introspective equilibrium, i.e., $\rho = \rho(\boldsymbol{u}^0) > \overline{\rho}$. As b increases, the dominance parameter decreases. Let $\overline{b}$ be the bonus for which the dominance parameter attains the value $\overline{\rho}$. Then,

$$\frac{1-\rho}{\rho} = \frac{u_{11} - u_{21}}{u_{22} - u_{12}} \qquad \text{and} \qquad \frac{1-\overline{\rho}}{\overline{\rho}} = \frac{u_{11} + \overline{b} - u_{21}}{u_{22} - u_{12}}. \tag{I.1}$$

The difference in value between the games with $b = 0$ (with coordination failure) and with $b = \overline{b}$ (with miscoordination) is

$$\begin{aligned}\Delta &= \overline{p_{11}}\,(u_{11} + \overline{b}) + \overline{p_{12}}\,u_{12} + \overline{p_{21}}\,u_{21} + \overline{p_{22}}\,u_{22} - u_{22} \\ &= \overline{p_{11}}\,(u_{11} + \overline{b} - u_{21}) - (\overline{p_{11}} + \overline{p_{12}})(u_{11} - u_{21}) + (\overline{p_{11}} + \overline{p_{12}})(u_{11} - u_{22}) - \overline{p_{12}}\,(u_{22} - u_{12}),\end{aligned}$$

where we have used $\overline{p_{12}} = \overline{p_{21}}$ and $\overline{p_{22}} = 1 - \overline{p_{11}} - \overline{p_{12}} - \overline{p_{21}}$. Applying (I.1) to the factors $u_{11} + \overline{b} - u_{21}$ and $u_{11} - u_{21}$ (and reordering terms) gives

$$\Delta = (\overline{p_{11}} + \overline{p_{12}})(u_{11} - u_{22}) - \left(\frac{\overline{p_{11}} + \overline{p_{12}}}{\rho} - \frac{\overline{p_{11}}}{\overline{\rho}}\right)(u_{22} - u_{12}).$$

Hence, $\Delta < 0$ if and only if

$$\frac{u_{11} - u_{22}}{u_{22} - u_{12}} < \frac{1}{\rho} - \frac{\overline{p_{11}}}{\overline{\rho}\,(\overline{p_{11}} + \overline{p_{12}})} = \frac{\rho^c - \rho}{\rho\,\rho^c}.$$

As the left-hand side is non-negative, Δ can be negative only if $\rho < \rho^c$. In that case, $\Delta < 0$ is equivalent to

$$\frac{u_{22} - u_{12}}{\rho} = \frac{u_{11} - u_{21}}{1 - \rho} > \frac{\rho^c}{\rho^c - \rho}\,(u_{11} - u_{22}),$$

where the equality follows from (I.1). □

II Collusion

We consider two other commonly-used models of collusion. In addition, because models of collusion have the structure of a social dilemma, we also study the classic repeated prisoner's dilemma. The results demonstrate that our results in Section 3.3.2 are robust.

II.1 Tourists and natives

This appendix studies the effects of a change in the competitiveness of the market as measured by the fraction of consumers who buy from the firm with the lowest price. We consider a simple model of price insensitive "tourists" and best-price shopping "natives," or, closer to most applications, loyal buyers and switchers.[1] There are two firms, labeled by $i \in \{1,2\}$. In each period $\tilde{t} = 0, 1, \ldots$, firms choose a price $p \in \{H, L\}$, with $H > L > 0$, and a mass of consumers (of measure 1) decides which firm to buy from. A fraction $m_s \in (0,1)$ of consumers are *switchers*: they buy from the firm with the lowest price. In addition, each firm i has a mass $m_\ell \in (0, \frac{1}{2})$ of *loyal buyers* (i.e., $m_s + 2m_\ell = 1$). The marginal cost of each firm is equal to 0, and we normalize by setting $L = 1$. Then, the payoffs in the one-shot game are given by

	H	L
H	$(m_\ell + \frac{1}{2}m_s)H, (m_\ell + \frac{1}{2}m_s)H$	$m_\ell H, m_\ell + m_s$
L	$m_\ell + m_s, m_\ell H$	$m_\ell + \frac{1}{2}m_s, m_\ell + \frac{1}{2}m_s$

As in Section 3.3.2, we assume that the one-shot game has the structure of a prisoner's dilemma (i.e., the low price L is a strictly dominant strategy for both firms). This is the case if and only if $H < 2(1 - m_\ell)$. Thus, in the absence of repetition, firms compete for the market.

For the repeated game, we again consider a collusive strategy and a cheating strategy. Under the collusive strategy, the firm chooses the high price H in every period as long as both firms chose the high price in all past periods; otherwise, it charges the low price L. Under the cheating strategy, the firm chooses the low price L in every period. Again, we identify the collusive strategy with s^1 and the cheating strategy with s^2. Then, using that $m_\ell + \frac{1}{2}m_s = \frac{1}{2}$ and $m_\ell + m_s = 1 - m_\ell$, the payoffs in the repeated game are given by

	s^1	s^2
s^1	$\frac{1}{2}H$	$(1-\delta)\, m_\ell H + \frac{1}{2}\delta$
s^2	$(1-\delta)(1-m_\ell) + \frac{1}{2}\delta$	$\frac{1}{2}$

where we have listed only the row player's payoffs. We again assume that collusion can be sustained as a subgame perfect equilibrium, i.e., $\delta > (2(1 - m_\ell) - H)/(1 - 2m_\ell)$. Then, the repeated game can be viewed as a coordination game, and coordination failure corresponds to both firms choosing the low price while miscoordination corresponds to one firm choosing the collusive strategy and the other firm choosing the cheating strategy.

The following result shows that, starting from a game with coordination failure, any change in payoff parameters that makes collusion less risky (i.e., decreases ρ) leads to a strict increase in the value even if it induces miscoordination:

[1] See Salop and Stiglits, 1977, Bargains and ripoffs: A model of monopolistically competitive price dispersion, *Review of Economic Studies*, 44, pp. 493–510.

Theorem II.1 (Collusion: Tourists & Natives). *Fix an introspective type space that satisfies Assumptions 1–5 and a game form such that there is coordination failure in introspective equilibrium. Then any change in payoff parameters that makes the dominance parameter smaller than $\overline{\rho}$ strictly increases the value of the game.*

Proof. As in Section 3.3.2, we consider a change of payoff parameters such that the dominance parameter decreases from $\rho > \overline{\rho}$ to $\rho' < \overline{\rho}$. Since $u_{11} > u_{22}$, the result clearly holds if the change in payoff parameters is such that firms choose the high price in introspective equilibrium (i.e., $\rho' < \underline{\rho}$). So suppose $\rho' \in [\underline{\rho}, \overline{\rho})$, and let δ and δ' be the discount factors in the games with dominance parameters ρ and ρ', respectively. Similarly, let m_ℓ and m'_ℓ be the fractions of loyal buyers for a firm in the games with dominance parameters ρ and ρ', respectively. Then, the inequality (13) in Lemma C.3 gives

$$\Delta > (p'_{11} + p'_{12})(1 - \delta')(\tfrac{1}{2} - m'_\ell).$$

The result then follows by noting that the right-hand side is positive. □

Examples of changes in the dominance parameter that make collusion less risky include an increase in the fraction m_ℓ of loyal buyers, an increase in the discount factor δ, and an increase in the high price H. Theorem II.1 shows that any of these changes makes firms better off.

II.2 Homogeneous goods

This appendix studies the limiting case of Section 3.3.2 where the two firms produce identical goods (i.e., $b = c$). To avoid problems with the nonexistence of a pure Nash equilibrium, we assume that firms can undercut each other only by a fixed amount $\frac{1}{2}\eta > 0$ (taken to be small, i.e., $\eta \ll a$). The model is otherwise the same as in Section 3.3.2: In the repeated game, the collusive strategy σ^* is to choose the monopoly price p^* in every period as long as both firms chose the monopoly price in all past periods, and to choose the competitive price $p^N = 0$ otherwise. The cheating strategy σ^c is to choose $p^c := p^* - \frac{1}{2}\eta$ in every period as long as both firms chose the monopoly price in all past periods, and to choose the competitive price $p^N = 0$ otherwise. We normalize and set $b = 1$. Then, if we again identify the collusive and the cheating strategy with s^1 and s^2, the payoffs in the repeated game are given by

	s^1	s^2
s^1	$\frac{1}{8}a^2$	0
s^2	$\frac{1}{4}(1-\delta)(a^2-\eta^2)$	$\frac{1}{8}(1-\delta)(a^2-\eta^2)$

where we have listed only the row player's payoffs. We again assume that collusion can be sustained as a subgame perfect equilibrium, i.e., $\delta > (a^2 - 2\eta^2)/(2a^2 - 2\eta^2)$. As before, the repeated game can be viewed as a coordination game, and coordination failure corresponds to both firms choosing the cheating strategy while miscoordination corresponds to one firm choosing the collusive strategy and the other firm choosing the cheating strategy.

The following result considers the effects of an increase in the discount factor δ:

Theorem II.2 (Collusion: Homogeneous Goods). *Fix an introspective type space that satisfies Assumptions 1–5 and let $\delta, \delta' \in \left((a^2 - 2\eta^2)/(2a^2 - 2\eta^2), 1 \right)$ be such that there is coordination failure in introspective equilibrium when the discount factor is δ (i.e, $\rho(\delta) > \overline{\rho}$) but not when the discount factor is δ' (i.e., $\rho(\delta') < \overline{\rho}$). Then, if either δ' is so large that $\rho(\delta') < \underline{\rho}$ or $\delta' - \delta$ is sufficiently small, the value of the game with discount factor δ' (with miscoordination) is strictly larger than the value of the game with discount factor δ (with coordination failure).*

Proof. Define $\rho := \rho(\delta)$ and $\rho' := \rho(\delta')$ to be the dominance parameters for the games with discount factors δ and δ', respectively. Again, the result clearly holds if δ' is so large that firms choose the collusive strategy in introspective equilibrium (i.e., $\rho' < \underline{\rho}$). So suppose $\rho' \in [\underline{\rho}, \overline{\rho})$. Let $\varepsilon := \delta' - \delta$ be the difference between the two discount parameters, and let $\overline{\delta}$ be the discount parameter for which $\rho(\overline{\delta}) = \overline{\rho}$. Substituting the payoffs into Eq. (13) from Lemma C.3 gives

$$\begin{aligned} 8\Delta &> (p'_{11} + p'_{12})(1 - \delta')(a^2 - \eta^2) - (\delta' - \delta)(a^2 - \eta^2) \\ &> (\overline{p_{11}} + \overline{p_{12}})(1 - \overline{\delta})(a^2 - \eta^2) - \varepsilon(\overline{p_{11}} + \overline{p_{12}})(a^2 - \eta^2) - \varepsilon(a^2 - \eta^2), \end{aligned}$$

(as in the proof of Theorem 3.5). It follows that $\Delta > 0$ if the difference ε between discount parameters is sufficiently small. ☐

II.3 Prisoner's dilemma

This appendix studies the classic prisoner's dilemma. In each period $\tilde{t} = 0, 1, \ldots$, players can either cooperate (play c) or defect (play d). The payoffs in the one-shot game are given by

	c	d
c	C, C	S, T
d	T, S	D, D

where $T > C > D > S$. In the repeated game, players choose between a cooperative strategy and always defect. Under the cooperative (grim trigger) strategy, the player cooperates in every period as long as both players cooperated in all past periods; otherwise, he defects. Under always defect, the player defects in every period. Again, we identify the cooperative strategy with s^1 and always defect with s^2. Then, the payoffs in the repeated game are given by

	s^1	s^2
s^1	C	$(1-\delta) S + \delta D$
s^2	$(1-\delta) T + \delta D$	D

where we have listed only the row player's payoffs. We again assume that cooperation can be sustained as a strict subgame perfect equilibrium, i.e., $\delta > (T - C)/(T - D)$. We can then view the repeated game as a coordination game, where coordination failure corresponds to both players choosing to always defect while miscoordination corresponds to one player trying to initiate cooperation (i.e., choosing s^1) and the other player choosing to always defect. It will be convenient to write (δ, C, S, T, D) for the game form $\boldsymbol{u}$ and to denote the corresponding dominance parameter by $\rho(\delta, C, S, T, D)$.

We consider the effect of two types of changes: An increase in the discount factor δ and a decrease in the payoff D when cooperation breaks down. Both types of changes make cooperation less risky (i.e., ρ decreases with δ and increases with D). Hence, if we start from a game with coordination failure, both these changes can induce miscoordination. The following result shows that, again, players are better off when there is miscoordination than if there is coordination failure:

Theorem II.3 (Prisoner's Dilemma). *Fix an introspective type space that satisfies Assumptions 1–5 and a game form* $\boldsymbol{u} = (\delta, C, S, T, D)$ *with* $\delta \in (\,(T - C)/(T - D), 1\,)$, *and let* $\delta' \in (\delta, 1)$ *and* $D' < D$.

(a) Suppose that there is coordination failure in introspective equilibrium when the discount factor is δ *but not when it is* δ' *(i.e.,* $\rho(\delta, C, S, T, D) > \overline{\rho}$ *and* $\rho(\delta', C, S, T, D) < \overline{\rho}$*). Then, the value of the game with the discount factor* δ' *is strictly larger than that of the game with the discount factor* δ.

(b) Suppose that there is coordination failure in introspective equilibrium when the defection payoff is D *but not when it is* D' *(i.e.,* $\rho(\delta, C, S, T, D) > \overline{\rho}$ *and* $\rho(\delta, C, S, T, D') < \overline{\rho}$*). Then, the value of the game when the defection payoff is* D' *is strictly larger than when the defection payoff is* D *if either* D' *is so large that* $\rho(\delta, C, S, T, D') < \underline{\rho}$ *or* $D - D'$ *is sufficiently small.*

Proof. It will be convenient to combine the proofs of (a) and (b) by considering the game forms $\boldsymbol{u} = (\delta, C, S, T, D)$ and $\boldsymbol{u}' = (\delta', C, S, T, D')$. Then, with some abuse of notation, we can take $\delta' > \delta$ and $D' = D$ for proving (a), and $\delta' = \delta$ and $D' < D$ for proving (b). Write ρ and ρ' for the dominance parameters associated with $\boldsymbol{u}$ and $\boldsymbol{u}'$, respectively. As before, if the change in payoff parameters is sufficiently large (i.e., $\rho' < \underline{\rho}$), the result follows from the fact that $u_{11} > u_{22}$. So suppose $\rho' \in [\,\underline{\rho}, \overline{\rho}\,)$. Then we can again apply Lemma C.3. In this case, Eq. (13) gives

$$\begin{aligned}\Delta &> (p'_{11} + p'_{12})(1 - \delta')(T - D') - (D - D') \\ &> (\,\overline{p_{11}} + \overline{p_{12}}\,)(1 - \delta')(T - D) + (\,\overline{p_{11}} + \overline{p_{12}}\,)(1 - \delta')(D - D') - (D - D').\end{aligned}$$

So, if $\delta' > \delta$ and $D' = D$, we have $\Delta > 0$, proving (a); and if $\delta' = \delta$ and $D' < D$, then $\Delta > 0$ provided D is sufficiently close to D', proving (b). □

III Asymptotic stability

This section shows that introspective equilibrium is asymptotically stable whenever the rank belief function $F(t \mid t)$ has a finite number of local extrema. Fix a game $\mathcal{G} = (\boldsymbol{u}, \mathcal{T})$, where $\mathcal{T} = (F, \tau^0)$, and let τ be the equilibrium threshold for the introspective equilibrium. That is, τ is the limit of the level-k thresholds τ^k as $k \to \infty$. For every $t \in T$ and $\varepsilon > 0$, let

$$B_\varepsilon(t) := \{t' \in T : |t - t'| < \varepsilon\}$$

be the ε-ball around t. We say that introspective equilibrium is *asymptotically stable* if it is both attracting and Lyapunov stable, i.e., if the equilibrium threshold τ satisfies the following conditions:

(ATTR) There is an $\varepsilon > 0$ such that for any $\tilde{\tau}^0 \in B_\varepsilon(\tau)$, the introspective process $\{\tilde{\tau}^k\}_k$ starting at $\tilde{\tau}^0$ converges to τ, i.e., $\lim_{k\to\infty} \tilde{\tau}^k = \tau$; and

(LYAP) For every $\eta > 0$, there is a $\delta > 0$ such that if $\tilde{\tau}^0 \in B_\delta(\tau)$, then the introspective process $\{\tilde{\tau}^k\}_k$ starting at $\tilde{\tau}^0$ remains in $B_\eta(\tau)$, i.e., $\tilde{\tau}^k \in B_\eta(\tau)$ for all $k \geq 0$.

The following result shows that introspective equilibrium is asymptotically stable for generic payoff parameters:

Proposition III.1 (Asymptotic Stability). *Under Assumptions 1–4, provided that the rank belief function $F(t \mid t)$ has only finitely many local extrema, introspective equilibrium is (generically) asymptotically stable in coordination games.*

Proof. We show that introspective equilibrium satisfies (ATTR) and (LYAP) except when $F(\tau^0 \mid \tau^0) = 1 - \rho$ or $F(\tau \mid \tau)$ is a local extremum. Since, by assumption, there are only finitely many values for ρ such that $F(\tau^0 \mid \tau^0) = 1 - \rho$ or $1 - \rho$ is a local extremum of $F(t \mid t)$, this establishes that introspective equilibrium is asymptotically stable for generic $\boldsymbol{u}$. So suppose that $F(\tau^0 \mid \tau^0) \neq 1 - \rho$ and that $F(\tau \mid \tau)$ is not a local maximum or minimum. We start with (ATTR). First suppose $\tau = 0$; the proof for $\tau = 1$ is similar and thus omitted. By Assumption 2, we have $\tau^0 > 0$, and by the proof of Proposition 2.1, $F(t \mid t) < 1 - \rho$ for all $t \in [0, \tau^0]$. Then, for any level-0 threshold $\tilde{\tau}^0 \in B_{\tau^0}(\tau)$, the introspective process converges to τ; so, the claim holds with $\varepsilon = \tau^0$. Next suppose $\tau \in (0, 1)$. We will construct a nonempty open interval $(\tau_{\min}, \tau_{\max})$ containing τ such that, starting from any level-0 threshold $\tilde{\tau}^0 \in (\tau_{\min}, \tau_{\max})$, the introspective process converges to τ; this shows that the claim holds with $\varepsilon := \min\{\tau - \tau_{\min}, \tau_{\max} - \tau\} > 0$. We prove the result for the case where $F(\tau^0 \mid \tau^0) < 1 - \rho$; the proof for the case when $F(\tau^0 \mid \tau^0) > 1 - \rho$ is similar and thus omitted. Again, by the proof of Proposition 2.1, $\tau^0 > \tau$, the level-k thresholds $\{\tau^k\}$ form a decreasing sequence that converges to τ, $F(\tau \mid \tau) = 1 - \rho$, and $F(t \mid t)$ is strictly smaller than $1-\rho$ for all t in the interval (τ, τ^0); so, we can set $\tau_{\max} = \tau^0$, where we note that $\tau_{\max} > \tau$. We next construct $\tau_{\min} < \tau$. By the continuity of $F(t \mid t)$ on $[0, \tau]$ (see the proof of Proposition 3.1), there is a $t \in [0, \tau]$ such that $F(t \mid t)$ is a local maximum. We define

$$\tau_{\min} := \sup\{t \in [0, \tau] : F(t \mid t) \text{ is a local maximum}\}.$$

Since, by assumption, $F(t \mid t)$ has a finite number of local extrema and $F(\tau \mid \tau)$ is not a local extremum, $\tau_{\min}$ is in fact strictly smaller than τ. Since $F(\tau \mid \tau) = 1 - \rho$, it follows that $F(t \mid t) > 1 - \rho$ for all $t \in (\tau_{\min}, \tau)$, so (again by the proof of Proposition 2.1) any introspective process $\{\tilde{\tau}^k\}_k$ starting at a level-0 threshold $\tilde{\tau}^0 \in (\tau_{\min}, \tau]$ converges to τ, as required.

The fact that introspective equilibrium satisfies (LYAP) now follows immediately. Fix $\eta > 0$, and let $\varepsilon > 0$ be as constructed in the proof of property (ATTR). By the argument above, for any $\tilde{\tau}^0 \in B_\varepsilon(\tau)$, the introspective process $\{\tilde{\tau}^k\}_k$ starting at $\tilde{\tau}^0$ has the property that $|\tau - \tilde{\tau}^k|$ decreases with k. Hence, the claim holds for $\delta := \min\{\eta, \varepsilon\}$. □